IEOR 8100-001: Learning and Optimization for Sequential Decision Making

Homework 1 solutions [1]

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Problem 1. Prove $e^{\sqrt{ln(r)}}$ is $o(r)$.

Proof. Note:

$$\begin{aligned}\lim_{r\to\infty}\frac{e^{\sqrt{ln(r)}}}{r} &= \lim_{r\to\infty}\frac{e^{\sqrt{ln(r)}}}{e^{ln(r)}}\\ &= \lim_{r\to\infty} e^{\sqrt{ln(r)}-ln(r)}\\ &= e^{\lim_{r\to\infty}\sqrt{ln(r)}-ln(r)} = e^{-\infty} = 0\end{aligned}$$

□

Problem 2. There is 75% chance of rain on Monday. There is 50% chance of rain on Tuesday if that it rains on Monday, and 20% otherwise. If it rains on any given day, there is a 70% chance that Alice will get wet on her way to work. (She does not change her behavior based on what happened the earlier day). What is the probability that Alice will get wet from rain on both Monday and Tuesday?

Solution . Let A be a random variable for Alice's state on Monday and Tuesday and D be a random variable representing the weather on Monday and Tuesday. Let $A \in \{WetWet, WetDry, DryWet, DryDry\}$ and $D \in \{RainRain, RainSun, SunRain, SunSun\}$ We want:

$$\begin{aligned}Pr(A = WetWet) &= E[\mathbf{1}\{A = WetWet\}]\\ &= E[E[\mathbf{1}\{A = WetWet\}|D]]\\ &= E[\mathbf{1}\{A = WetWet\}|D = RainRain]Pr(D = RainRain)\end{aligned}$$

Note Pr(D=RainRain) = Pr(Monday = Rain)*Pr(Tuesday = Rain | Monday = Rain) = .75 * .5 = 3/8. Also note Pr(A = WetWet | D = RainRain) = $.7^2$. Thus

$$Pr(A = WW) = .7^2 * .75 * .5 = .18375 \quad (1)$$

Problem 3. A standard 52-card deck [2] is randomly partitioned into four 13-element sets, which are dealt to players named Alice, Bob, Tom, and Harry.

[2] A standard 52-card deck is the set $\{2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king, ace\} \times \{clubs, diamonds, hearts, spades\}$.

(a) Calculate Pr(Alice gets exactly 2 aces|Bob gets exactly 1 ace).
Hint: Use $\Pr(A|B) = \Pr(A \cap B)/\Pr(B)$.

(b) Let C and S denote the number of clubs and spades, respectively, dealt to Alice. Calculate $E(C|S)$ as a function of S.
Hint : Observe that given the number of spades, the number of clubs, hearts, diamonds dealt to Alice have the same conditional distribution.

Solution a. Note $\Pr(\text{B gets 1 ace}) = \frac{\#\text{ways to get 1 ace hand}}{\#\text{ways to get hand}}$. This probability is:

$$\binom{4}{1}\binom{51}{12}/\binom{52}{13} \tag{2}$$

since there are 4 ways to get the ace and then 51 choose 12 ways to get the rest of the hand all over 52 choose 13 ways to get an arbitrary hand. Similarly Pr(B gets 1 ace $\cap$ A gets 2 aces) is:

$$\binom{4}{1}\binom{51}{12}\binom{3}{2}\binom{52-13-2}{11}/\left(\binom{52}{13}\binom{52-13}{13}\right) \tag{3}$$

Since once we've drawn B's hand, we have 52-13 cards left to deal from and 3 remaining aces. Thus 3 choose 2 ways to distribute them to A and 52-13-2 choose 11 ways to deal the rest of the hand.

Solution b. Let H be the number of hearts dealt to Alice and D be the number of diamonds. Now by symmetry:

$$E[C|S] = E[D|S] = E[H|S] \tag{4}$$

and note since Alice is dealt 13 cards

$$E[C|S] + E[D|S] + E[H|S] = 13 - S \tag{5}$$

which implies

$$E[C|S] = (13 - S)/3 \tag{6}$$

Problem 4. Let $X_1, X_2, \ldots, X_n$ be independent uniformly-distributed random samples from the interval $[0, 1]$. Define the following probabilities:

- $p(n)$ is the probability that $\min_k X_k > 0.01$.
- $q(n)$ is the probability that $\min_{i \neq j} |X_i - X_j| \leq \frac{1}{n^2}$.
- $r(n)$ is the probability that $\min_{i \neq j} |X_i - X_j| > \frac{1}{100n}$.
- $s(n)$ is the probability that exactly $\lfloor n/2 \rfloor$ of the numbers $X_1, ..., X_n$ lie in $[0, \frac{1}{2}]$.

Estimate the asymptotic behavior of each of these probabilities as n tends to infinity. Specifically, for each of the sequences $p(n), q(n), r(n), s(n)$, determine whether the sequence

(A) tends to zero exponentially fast, i.e. is $O(c^n)$ for some constant $c < 1$;

(B) tends to zero, but not exponentially fast;

(C) remains bounded away from 0 and 1;

(D) tends to 1.

You do not need to justify your answer. Just answer A, B, C, or D for each of the sequences.

Solution a. Tends to 0 exponentially fast. To see this note

$$Pr(\min_k X_k > .01) = Pr(\cap_{i1}^n (X_i > .01)) = \prod_{i1}^n Pr(X_i > .01) = .99^n \tag{7}$$

where the second equality follows by independence and the third by uniformity.

Solution b. Remains bounded away from 0 to 1. To see this fix an $\epsilon > 0$, now we'll call a set of realized points $\{x_1, \ldots, x_n\}$ ϵ-independent if the intervals $[x_i, x_i + \epsilon]$ are all pairwise disjoint. Note $1 - q(n) = Pr(\{X_i\}_{i=1}^n n^{-2}\text{-independent})$ and thus bounding the probability of the sample being n^{-2}-independent away from 1 is equivalent to bounding away q(n) from 0. Now:

$$\begin{aligned}
&Pr(\{X_i\}_{i=1}^n \epsilon\text{-independent}) \\
&\quad = \prod_{k=2}^n Pr(\{X_i\}_{i=1}^k \epsilon\text{-independent} | \{X_i\}_{i=1}^{k-1} \epsilon\text{-independent}) \\
&\quad \le \prod_{k=2}^n Pr(X_k \notin \cup_{i=1}^{k-1} [x_i, x_i + \epsilon] | \{X_i\}_{i=1}^{k-1} \epsilon\text{-independent}) \\
&\quad \le \prod_{k=2}^m (1 - (k-2)\epsilon) \le \prod_{k=2}^n e^{-(k-2)\epsilon} \\
&\quad = e^{-(n-1)(n-2)\epsilon/2}
\end{aligned}$$

where the first equality follows inductively by conditioning. The first inequality is true inequality because intervals may press against the sides, that is: $|[x_i, x_i+\epsilon] \cup [0,1]| \le |[x_i, x_i+\epsilon]|$. The second inequality follows by uniformity. Plugging in $\epsilon = n^{-2}$ shows the probability is bounded away from 1.

Now to see the probability is bounded away from 1 note for any i, j that $\Pr(|x_i - x_j| \le \frac{1}{n^2}|x_i) = n^{-2} + \min\{x_i, n^{-2}, 1 - x_i\}$ since the probability should be the chance of falling into an interval of length $2/n^2$ centered at x_i barring collisions with the "edges". Integrating over x_i gives:

$$Pr(|x_i - x_j| \le \frac{1}{n^2}) = 2n^{-2} - n^{-4} \tag{8}$$

there are $\binom{n}{2}$ ways such an event could happen, so in expectation the number of violations is $\binom{n}{2}(2n^{-2} - n^{-4})$ which tends to 1 as $n \to \infty$. Lets call the random variable that counts the number of violations V. Note as n tends to infinity:

$$1 = E[V] = \lim_{n\to\infty} \sum_{i=1}^n Pr(V > n) = q(n) + Pr(Y > 1) \tag{9}$$

thus if we can show $Pr(Y > 1) > 0$ in limit we can conclude q(n) is bounded away from 1. But recall we've shown the probability of one violation in set of size n is bounded away from 0, to show the the probability of two violations is bounded away from 0 (and thus $\Pr(Y > 1)$ bounded away from 0 which would imply q(n) bounded away from 1) apply the argument above on the sets $A = \{X_i\}_{i=1}^{n/2}$ and $B = \{X_i\}_{i=n/2}^{n}$. Note that Y_A and Y_B, the indicator random variables for violations in A and violations in B respectively are independent and bounded away from 0. Thus the probability of both occurring is bounded away from 0 (simply the product) which, finally, implies q(n) is bounded away from 1.

Solution c. Tends to 0 exponentially fast. To see this note the question is equivalent to ask if $Pr(\{X_i\}_{i=1}^{n}$ is $\frac{1}{100n}$-independent. Now use the estimate given in the first part of (b) with $\epsilon = \frac{1}{100n}$, which immediately implies that the probability tends to 0 at rate roughly e^{-n}.

Solution d. Tends to 0 but not exponentially fast. To see this, note X_i's fall in either $[0, .5]$ or $[.5, 1]$ with probability 1/2 and there are $\binom{n}{floor(n/2)}$ ways to fall into the two sets equally. Each realization has probability 2^{-n}. Thus:

$$s(n) = \binom{n}{floor(n/2)} * 2^{-n} \tag{10}$$

recall Stirlings approximation, $n! \approx \sqrt{2\pi n}(n/e)^n$ which implies $\binom{n}{floor(n/2)} \approx n^{-1/2}2^n$ (up to constant factors) which implies $s(n)$ goes to 0 at rate $\frac{1}{\sqrt{n}}$.

Problem 5. (a) You are selling your bike. You get offers one by one. You have decided to stop and accept an offer as soon as you see one which is better than the first offer you got. (You always skip the first offer). What is the expected number of offers you will to wait for, including the first one, until you accept an offer? Mathematically, let's model this process as follows. Let X_1, X_2, ... denote an infinite sequence of independent uniformly-distributed random samples from the interval $[0, 1]$. (Interpretation: X_i is the i^{th} offer.) Let τ be the smallest $i > 1$ such that $X_i > X_1$. What is $E[\tau]$?

(b) Now suppose that you modify your stopping rule. For some fixed predetermined number k, you do not accept any of the first k offers. Let h be the second best offer observed among the first k. Your policy is to accept the next offer (after the first k) which is better than h. In more precise terms, let X_1, X_2, . . . be a sequence of independent random variables uniformly distributed in $[0, 1]$ as before, let X_a, X_b be the two largest elements of the set $X_1, \ldots, X_k$, and let ρ be the smallest $i > k$ such that $X_i > X_b$. What is $E(\rho)$, as a function of k?

Hint: Use following formula for expected value of a non-negative integer valued random variable Y: $E[Y] = \sum_{n=0}^{\infty} \Pr(Y > n)$.

Solution a. Imagine $X_1 = x \in [0, 1]$, then each subsequent draw has probability $1 - x$ of being larger than x and the time until we draw something larger than x is a geometric random variable with success probability $1 - x$. So by conditioning on the value of your first draw:

$$
\begin{aligned}
E[T] &= E[E[T|X_1]] \\
&= 1 + \int_0^1 E[E[T|X_1 = x]]Pr(X_1 = x) \\
&= 1 + \int_0^1 \frac{1}{1-x} = \infty
\end{aligned}
$$

so in expectation our time until stopping is infinite.

Solution b. For this part first recall the following elementary fact about the order statistics of iid uni[0,1] random variables, namely that the pdf of the kth order statistic of n draws is:

$$
f_{(k)}(x) = nf(x)\binom{n-1}{k-1}F(x)^{k-1}(1-F(x))^{n-k} \tag{11}
$$

we're concerned with the $k-1^{st}$ order statistic from k draws, thus using the formula

$$
f_{[k-1:k]}(x) = k(k-1)x^{k-2}(1-x) \tag{12}
$$

now like before we'll condition on the value of our stopping criteria:

$$
\begin{aligned}
E[T] &= E[E[T|X_{[k-1:k]}]] \\
&= k + \int_0^1 E[E[T|X_{[k=1:k]} = x]]Pr(X_{[k-1:k]} = x) \\
&= k + \int_0^1 k(k-1)x^{k-2}(1-x)\frac{1}{1-x} \\
&= k + k(k-1)\int_0^1 x^{k-2} = 2k
\end{aligned}
$$